

A note on Weyl transformations in two-dimensional dilaton gravity*

Marco Cavaglià ^{†‡}

Center for Theoretical Physics

Laboratory for Nuclear Science and Department of Physics

Massachusetts Institute of Technology

77 Massachusetts Avenue, Cambridge Massachusetts 02139, USA

and

Istituto Nazionale di Fisica Nucleare, Sede di Presidenza, Roma, Italia.

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Abstract

We discuss Weyl (conformal) transformations in two-dimensional matterless dilaton gravity. We argue that both classical and quantum dilaton gravity theories are invariant under Weyl transformations.

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Introduction

This letter deals with Weyl transformations in two-dimensional dilaton gravity. Motivated by some recent papers about the role of Weyl transformations

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[†]Post office address: Massachusetts Institute of Technology, Marco Cavaglià 6-408, 77 Massachusetts Avenue, Cambridge MA 02139-4307, USA.

[‡]E-mail: cavaglia@mitlns.mit.edu

in two-dimensional dilaton gravity (see e.g. [1, 2]), we want to answer the following question: Is two-dimensional matterless dilaton gravity invariant under Weyl conformal transformations? A short answer to this question seems to be negative [1, 2]. However, a more careful analysis shows that the *physical* properties of classical and quantum dilaton gravity theories are actually invariant under Weyl rescalings of the metric. This paper provides evidence in support of this claim.

The curse of Weyl transformations

Our starting point is the action

$$S_{DG} = \int_{\Sigma} d^2x \sqrt{-\gamma} \left[\phi R^{(2)}(\gamma) + \mathcal{V}(\phi) - \frac{d}{d\phi} [\ln |\mathcal{W}(\phi)|] (\nabla\phi)^2 \right], \quad (1)$$

where $\mathcal{V}(\phi)$ and $\mathcal{W}(\phi)$ are functions of the dilaton ϕ and $\gamma_{\mu\nu}$ is a two-dimensional metric with hyperbolic signature. The interest in the two-dimensional dilaton gravity theories (1) is motivated by their relation to a number of (physical) N -dimensional spacetimes, such as black holes and p-branes, via compactification of $N - 2$ dimensions. The most remarkable example is the N -dimensional spherically symmetric black hole which is described by an effective two-dimensional theory (1) upon dimensional reduction by integration on the spherical coordinates [3, 4].

As is well-known [5, 6], a classical Weyl transformation

$$\gamma_{\mu\nu}(x) = \tilde{\gamma}_{\mu\nu}(x) \frac{1}{\Omega(\phi)}, \quad (2)$$

where $\Omega(\phi)$ is a generic function of the dilaton, is often used to simplify Eq. (1). Choosing $\Omega(\phi) = 1/\mathcal{W}(\phi)$ the kinetic term of the dilaton in the action (1) can be set to zero. The action becomes

$$\tilde{S}_{DG} = \int_{\Sigma} d^2x \sqrt{-\tilde{\gamma}} \left[\phi \tilde{R}^{(2)}(\tilde{\gamma}) + \tilde{\mathcal{V}}(\phi) \right], \quad (3)$$

where

$$\tilde{\mathcal{V}}(\phi) = \frac{\mathcal{V}(\phi)}{\Omega(\phi)} = \mathcal{V}(\phi) \mathcal{W}(\phi). \quad (4)$$

Many authors consider Weyl transformations to be legitimate in classical dilaton gravity theories [5, 6]. However, some concerns have been raised about

their use in quantum theory [1, 2]. The main objection to using Weyl transformations (2) is that Weyl-related theories may describe locally inequivalent theories and change the global structure of the theory. An example which is frequently found in the literature is dilaton gravity with constant dilaton potential in the twiddle frame (3), also known as the Callan-Giddings-Harvey-Strominger (CGHS) matterless model [7]

$$\tilde{S}_{CGHS} = \int_{\Sigma} d^2x \sqrt{-\tilde{\gamma}} [\phi \tilde{R}^{(2)}(\tilde{\gamma}) + 4\lambda^2]. \quad (5)$$

In the untwiddle frame the CGHS model is described by Eq. (1) with $\mathcal{V}(\phi) = 4\lambda^2\phi$ and $\mathcal{W}(\phi) = \phi^{-1}$. The two frames are related by the Weyl transformation (2) with $\Omega(\phi) = \phi$. Varying Eq. (5) with respect to ϕ it is straightforward to prove that the CGHS model in the twiddle frame describes a flat Minkowski spacetime. This is not true in the untwiddle frame. So both local properties and global structure of the spacetime in the two frames are different. This fact has often been used to support the claim that models that are related by Weyl transformations describe, generally, dynamically inequivalent theories.

The rescue of Weyl transformations

The conclusion that we have reached in the previous section is inaccurate, both from classical and quantum points of view.

Let us consider first the classical theory. While it is correct to say that the twiddle and untwiddle theories are dynamically nonequivalent when considered separately, the two theories have the same classical, i.e., *physical*, content *if* the Weyl transformation (2) which relates the two frames is properly taken into account. We have the following

Theorem. Classical two-dimensional pure dilaton gravity theory is invariant under the (Weyl) transformation

$$\mathcal{V}(\phi) = \tilde{\mathcal{V}}(\phi)\Omega(\phi), \quad \mathcal{W}(\phi) = \tilde{\mathcal{W}}(\phi)/\Omega(\phi), \quad \gamma_{\mu\nu} = \tilde{\gamma}_{\mu\nu}/\Omega(\phi). \quad (6)$$

where $\Omega(\phi)$ is an arbitrary function of the dilaton.

Proof. Consider the action (1). Under the transformation (6) the Lagrangian transforms as

$$\mathcal{L} = \tilde{\mathcal{L}} + \sqrt{-\tilde{\gamma}} \tilde{\nabla} [\phi \tilde{\nabla}(\ln |\Omega|)]. \quad (7)$$

Therefore, the transformation (6) does not affect the equations of motion and is a symmetry of the model. This property can also be directly checked by implementing the transformation (6) into the equations of motion

$$\begin{aligned} \nabla_{(\mu} \nabla_{\nu)} \phi &- g_{\mu\nu} \nabla^2 \phi + \frac{1}{2} g_{\mu\nu} V(\phi) + \\ &+ \frac{d}{d\phi} \ln |\mathcal{W}(\phi)| [\nabla_{(\mu} \phi \nabla_{\nu)} \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2] = 0, \end{aligned} \quad (8)$$

$$R^{(2)}(g) + 2\nabla^2 \ln |\mathcal{W}(\phi)| - (\nabla \phi)^2 \frac{d^2}{d\phi^2} \ln |\mathcal{W}(\phi)| + \frac{d}{d\phi} \mathcal{V}(\phi) = 0. \quad (9)$$

It is straightforward to check that the transformation (6) leaves invariant Eq. (8) and Eq. (9). [Hint: to prove the invariance of Eq. (9) use the trace of Eq. (8).]

So the two theories are physically equivalent at classical level. Using the invariance (6) one can always set the dilatonic potential $\mathcal{V}(\phi)$ or the dilatonic kinetic coupling $\mathcal{W}(\phi)$ to a given function. The choice of $\Omega(\phi)$ coincides with a “gauge fixing” for the symmetry, Eq. (6). Clearly, the local properties of the gauge-fixed metric (e.g. the curvature) generally depend on the particular gauge that has been chosen. However, only quantities that are invariant under the symmetry (6) should be considered. It is particularly instructive to discuss this point in the context of dimensionally reduced models. Let us consider the N -dimensional Einstein-Hilbert action

$$S^{(N)} = \frac{1}{16\pi l_{pl}^{N-2}} \int d^N y \sqrt{-g} R^{(N)}(g). \quad (10)$$

As we mentioned above for spherically symmetric configurations

$$ds_N^2 = \gamma_{\mu\nu}(x) dx^\mu dx^\nu + G[\phi(x)] d\Omega_{N-2}^2, \quad (11)$$

Eq. (10) can be cast in the form (1) upon integration on the angular coordinates. Let us now choose a different ansatz for the N -dimensional metric,

$$d\tilde{s}_N^2 = \frac{\tilde{\gamma}_{\mu\nu}}{\Omega(\phi)} dx^\mu dx^\nu + G[\phi(x)] d\Omega_{N-2}^2. \quad (12)$$

This ansatz is related to the first one by the “conformal” redefinition of the two-dimensional metric field $\gamma_{\mu\nu} = \tilde{\gamma}_{\mu\nu}/\Omega(\phi)$. The dimensionally reduced action which is obtained by imposing the ansatz (12) is still of the form (1),

where $\tilde{\mathcal{V}}(\phi)$ and $\tilde{\mathcal{W}}(\phi)$ are related to $\mathcal{V}(\phi)$ and $\mathcal{W}(\phi)$ by Eq. (6). Note that from the N -dimensional point of view the two-dimensional Weyl symmetry (6) is just a field redefinition. Although the local properties of the metrics $\gamma_{\mu\nu}$ and $\tilde{\gamma}_{\mu\nu}$ are different, the *physical* properties of the N -dimensional system must be independent from the ansatz that has been chosen, i.e. they do not depend on the two-dimensional conformal frame which is being used. Therefore, only quantities which are invariant under the symmetry (6) make physically sense.¹

Now, let us turn to the quantum theory. Two-dimensional pure dilaton gravity is a general covariant, constrained, theory which is invariant under coordinate reparametrization. The theory possesses two degrees of freedom (the dilaton and a single gravitational degree of freedom that can be identified with the conformal factor of the metric) and two constraints, so it is actually a topological theory with no propagating degrees of freedom. Moreover, the constraints can be solved and the central term can be made vanishing by a suitable choice of the vacuum [9]. The whole physical content of the theory is given by the gauge invariant observables of the system. Because of the topological nature of two-dimensional dilaton gravity, the observables coincide with the conserved charges.

For theories described by the action (1) we have the single gauge invariant quantity (see [3],[6],[10]-[12] and references therein)

$$M = N(\phi) - \mathcal{W}(\phi)(\nabla\phi)^2, \quad N(\phi) = \int^\phi d\phi' [\mathcal{W}(\phi')\mathcal{V}(\phi')]. \quad (13)$$

The quantity M is gauge invariant and locally conserved. Apart from a constant normalization factor, for asymptotically flat geometries M coincides on-shell with the ADM mass of the system. Moreover, M is classically invariant under the Weyl symmetry (6) [8]. This can be proved by direct checking or by noticing that the dilaton action (1) can be rewritten as a function of M and ϕ as [3]

$$S_{DG} = \int_\Sigma d^2x \sqrt{-g} \frac{\nabla_\mu \phi \nabla^\mu M}{N(\phi) - M} + \text{surface terms}. \quad (14)$$

Since Eq. (1) is invariant under the transformation (6), and both ϕ and the Weyl combination $\sqrt{-g}g^{\mu\nu}$ are Weyl invariant, M must necessarily be invariant under Eq. (6).

¹Actually, the ADM mass, the temperature, and the flux of Hawking radiation turn out to be invariant under the Weyl transformation, Eq. (6) [8].

The quantity M is the only conserved charge of the theory and must determine completely the latter. Indeed, solving the constraints the effective gauge fixed action on the constraint shell ($\pi_\phi = 0$, $M' = 0$) is [3, 13]

$$S_{eff} = \int d\tau \left[\frac{dm}{d\tau} p_m - m \right], \quad (15)$$

where $m = M|_{boundary}$, p_m is the conjugate momentum of m , and τ is the proper time on the boundary. In the classical regime the physical content is completely determined by the value of m . In the quantum regime, the Hilbert state of the theory is completely determined by the eigenstates of the operator \hat{m} [3, 13]. This is the quantum generalization of the no-hair theorem for a classical spherically symmetric black hole in vacuo: A state is determined uniquely by the locally conserved charge (the ADM mass) of the system. Since the quantity M is invariant under the transformation (6), quantum two-dimensional dilaton gravity do not depend on the particular Weyl frame that has been chosen: Different frames lead to the same \hat{m} . Let us stress that the action (1) can be cast in the form (15) for any choice of $\Omega(\phi)$. Therefore, since M is invariant under the symmetry (6), and determines completely the Hilbert space of the theory, the latter do not depend on the Weyl frame. A possible objection to this statement could be that \hat{m} is a rather special operator and that, in general, operators corresponding to quantities which are not classically invariant under the transformation (6) are affected by the choice of the Weyl frame. Consequently, the quantum theory itself should depend on the Weyl gauge fixing. While the first part of this objection is correct, the conclusion is not. Indeed, since we are dealing with a constrained theory, only gauge invariant operators make sense. Since \hat{m} is the only gauge and Weyl invariant operator of the theory, any operator which is not Weyl invariant is necessarily not gauge invariant, so it does not have physical interpretation. This conclusion is also obtained through a different approach to dilaton gravity which has been worked in detail for the (matterless) CGHS model [14]. The essence of this approach is that the CGHS model (5) can be rewritten in terms of a couple of free fields [9] which are pure gauge. Once more, the only physical quantity of the theory coincides with the zero mode of the gauge and Weyl invariant mass operator.

In this letter we have seen that both classical and quantum pure two-dimensional dilaton gravity are unaffected by Weyl transformations. For the classical theory we have shown that Weyl transformations define a symmetry

of the system: The equations of motion are invariant under Weyl transformations. The choice of the Weyl frame is analogous to a choice of gauge fixing. The quantum theory of two-dimensional dilaton gravity is also *physically invariant* under Weyl transformations, in the sense that the Hilbert space is completely determined by the eigenstates of a (single) gauge and Weyl invariant observable. Let us finally stress that these results do not hold for matter-coupled two-dimensional dilaton gravity. For instance, if we couple the Polyakov action to the model (1) the resulting theory is not topological, the constraints cannot be solved, quantum anomalies appear and Weyl invariance is generally lost². Therefore, Weyl transformations are likely to play a very different role in matter-coupled dilaton gravity.

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²The quantization of the matter-coupled model has been thoroughly investigated in the literature for the CGHS case and several quantization approaches have been proposed to deal with anomalies. (See [9, 15] and references therein.)

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